

Beam Instabilities in the Scale Free Regime

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The instabilities arising in a one-dimensional beam sustained by the diffusive photorefractive nonlinearity in out-of-equilibrium ferroelectrics are theoretically and numerically investigated. In the “scale-free model”, in striking contrast with the well-known spatial modulational instability, two different beam instabilities dominate: a defocusing and a fragmenting process. Both are independent of the beam power and are not associated to any specific periodic pattern.

Introduction — Instability drives optical nonlinear waves into new and unexpected regimes. As for emergent behaviors in complex systems, so instability can lead a simple and featureless input wave to a whole spectrum of effects that range from apparently random fragmentation to highly regular pattern formation. Systems supporting solitons manifest a very general form of instability, modulational instability (MI), which fragments beams into patterns of periodic spots.[1–7] The entire process is rooted in the presence of one dominant spatial scale that forms in the nonlinear interaction, and this scale drives sufficiently wide waves into periodic patterns with a precise spatial period.

Recently, the observation of a novel class of optical spatial solitons [8, 9] has been reported in disordered out-of-equilibrium ferroelectrics [10, 11]. In this kind of media, the diffusion-driven photo-refractive nonlinearity [12] can be largely enhanced to support “scale-free” non-diffractive beams, which can have arbitrary amplitude and waist. The mechanism is a direct by-product of the photorefractive band-transport model in disordered systems, and applies to all photorefractive crystals, such as SBN and BaTiO₃, that are previously depoled into ferroelectric clusters (see, for example, [13]), alongside organic photorefractive glass [14] and photorefractive polymers [15]. More generally, scale-free optical effects stem from the interplay between diffraction and diffusion that mingle to cancel out their respective and intrinsic spatial scales, a condition that can occur in any wave supporting systems where diffusion and diffraction both play a relevant role. One known example is in slow-light experiments, when atomic diffusion and paraxial diffraction accurately cancel each other out, depriving the wave phenomenon of a characteristic scale and cancelling diffraction [16].

One-dimensional (stripe-shaped) scale-free solitary waves are observed to undergo a form of instability that breaks up [10], a process reminiscent of MI. The puzzling fact is that MI is wholly unexpected in a scale-free system, since, as we show below, the absence of a characteristic spatial scale provides no means for the propagation to select a dominant perturbation. Remarkably, in agreement with experiments, we find that the scale-free

regime, while curtailing the effects of MI, triggers entirely new beam instabilities.

We theoretically investigate the dynamics of Gaussian scale-free one-dimensional solitons in two-dimensional propagation, and consider the spatial evolution of a periodic perturbation with a transverse size different from that of the pump stripe beam. Although we find perturbations that can exponentially grow, these depend solely on pump beam waist and not on pump power, and no preferential spatial period emerges with maximum gain.

We stress that the relevant one-dimensional nonlinear optical model (i.e., when neglecting one transverse dimension, as commonly done in MI theory), does not admit an unstable regime, and the considered beam breaking is the result of a subtle multidimensional coupling in the scale-free model, such that also the instabilities retain the scale-free character in the fact that no preferential spatial scale is developed during the dynamics, a feature potentially useful in imaging applications [17].

Scale-free self-trapped beams — In the paraxial approximation, for a linearly polarized beam with wavelength λ , the slowly-varying optical field A ($|A|^2 = I$ is the optical intensity) obeys the nonlinear propagation equation (hereafter denoted as the *the scale-free model*) [10, 11]

$$2ik\frac{\partial A}{\partial z} + \nabla_{\perp}^2 A - \frac{L^2}{\lambda^2} \frac{(\partial_x I)^2 + (\partial_y I)^2}{4I^2} A = 0, \quad (1)$$

with $k = 2\pi n_0/\lambda$, n_0 the bulk refractive index, and the characteristic length L given by

$$L = 4\pi n_0^2 \epsilon_0 \sqrt{g} \chi_{PNR} (K_B T/q), \quad (2)$$

where K_B is the Boltzmann constant, T the temperature, q the elementary charge, ϵ_0 the vacuum permittivity, χ_{PNR} the low-frequency permittivity due to the *polar nano-regions* (PNR), and $g > 0$ the relevant component of the quadratic electro-optic tensor.

As has been previously shown [10, 11], Eq.(1), admits bell-shaped soliton solutions as long as $L \geq \lambda$. We specifically consider a perturbed striped beam. We begin writing Eq.(1) in dimensionless units as

$$i\frac{\partial \psi}{\partial z} + \frac{1}{2}\nabla_{xy}^2 \psi - \sigma \frac{(\partial_x |\psi|^2)^2 + (\partial_y |\psi|^2)^2}{|\psi|^4} \psi = 0, \quad (3)$$

where $x = X/W_0$, $y = Y/W_0$ and $z = Z/Z_0$, with $Z_0 = kW_0^2/2$ the diffraction length, W_0 an arbitrary beam waist, and $\sigma = L^2/8\lambda^2$, with $\sigma = 1/8$ for $L = \lambda$. In (3) $\psi = A/A_N$ is the dimensionless field scaled by an arbitrary factor A_N resulting from the intensity-independent feature of the scale-free model.

The solitary waves of Eq.(3) represent diffraction-free beams with arbitrary intensity and waist [10, 11]. Here we note that Eq.(3) also admits plane wave solutions for $\psi = A_0 = \text{constant}$, which, at variance with Kerr media, are not subject to any nonlinearly induced phase-shift (i.e., the nonlinear correction to the wave-vector is zero). We then consider the diffraction-free stripe (i.e., y -independent) solutions attained for $\sigma = 1/8$:

$$\psi_s(x, z) = \psi_0(x) \exp(i\beta z) = A_0 \exp\left(-\frac{x^2}{w_0^2}\right) \exp(i\beta z), \quad (4)$$

with $\psi_0(x) = \exp(-x^2/w_0^2)$, $\beta = -1/w_0^2$, and A_0 and w_0 arbitrary independent parameters, such that this self-trapped beam exists at any intensity level and for any waist (within the validity of the paraxial approximation). Note that as $w_0 \rightarrow \infty$, $\beta \rightarrow 0$, one recovers the plane-wave solution $\psi_0 = A_0$ mentioned above. Solutions also exist for $\sigma > 1/8$ ($L > \lambda$) [10, 11], but here we will focus on the case $L \cong \lambda$ ($\sigma = 1/8$), as this is the condition achieved in experiments.

Absence of Modulational Instability — We consider perturbations to the exact solution, which is written as

$$\psi(x, y, z) = [\psi_0(x) + p(x, y, z)] \exp(i\beta z). \quad (5)$$

The linearized evolution equation for the perturbation p reads as

$$i\partial_z p + \frac{1}{2}\nabla_{xy}^2 p - 4\sigma \left[\frac{\psi_{0x}^2}{\psi_0^2} p + \psi_{0x} \left(\frac{p+p^*}{\psi_0} \right)_x \right] = \beta p. \quad (6)$$

From Eq.(6), one sees that if $\psi_0 = A_0$ ($w_0 \rightarrow \infty$) no instability is expected (i.e., no solution such that p exponentially grows with z); the scale-free model of Eq.(3) is not exhibiting standard MI. This result is in striking contrast with the well-known fact that MI always accompanies the existence of solitary waves solutions, as for Kerr, saturable or quadratic nonlinearities. [8, 9] Conversely, in the scale-free model, one has self-trapped bright beams, but no instability for the plane-wave solutions; unaffected by MI, the scale-free system turns out to be fertile ground for very different forms of instabilities, as we discuss below.

Theory of the scale-free instabilities — We first note that Eq.(6) does not imply an exponential growth for a perturbation whose spatial profile in the x direction is the same as the pump beam ψ_0 , i.e., for p such that

$$p(x, y, z) = \psi_0(x)\alpha_+(z)e^{ik_y y} + \psi_0(x)\alpha_-(z)^*e^{-ik_y y}. \quad (7)$$

Indeed the term containing p^* in Eq.(6) disappears, so that there is no instability. Note that as Eq.(6) does not

contain coefficients explicitly dependent on y , p can be expressed as a plane wave expansion with respect to y without loss of generality. We then write the perturbation as

$$p(x, y, z) = \psi_1(x)\alpha_+(z)e^{ik_y y} + \psi_1(x)\alpha_-(z)^*e^{-ik_y y}, \quad (8)$$

with $\psi_1(x)$ an arbitrary spatial profile (specified below), different from $\psi_0(x)$. To keep the treatment as simple as possible, we limit analysis to the Gaussian soliton for $L = \lambda$ ($\sigma = 1/8$), and we take the profile for the perturbation $\psi_1(x)$ as a Gaussian with waist different from that of the pump $\psi_0(x)$, i.e.,

$$\psi_1(x) = \left(\frac{2}{\pi w_1^2}\right)^{1/4} \exp\left(-\frac{x^2}{w_1^2}\right), \quad (9)$$

such that $(\psi_1, \psi_1) = 1$, with the scalar product $(a, b) = \int a^* b dx$. We use Eq.(8) in Eq.(6) and project over ψ_1 , which corresponds to making an expansion in a Hermite-Gauss basis with respect to x and only retaining the first term of the expansion. We find the coupled equations for the amplitudes α_{\pm} after Eq.(8):

$$\pm 2i\frac{d\alpha_{\pm}}{dz} + \left(-k_y^2 + \frac{1}{w_0^2} - \frac{1}{w_1^2}\right)\alpha_{\pm} + \frac{w_1^2 - w_0^2}{w_0^4}\alpha_{\mp} = 0. \quad (10)$$

Note that the last term coupling α_{\pm} in Eq.(10) is responsible for the instabilities and is proportional to $w_1^2 - w_0^2$, hence for $w_1 = w_0$, we recover the result stated above, i.e., the absence of instability for a perturbation with the same x -size of the pump beam. Analogously, the instability disappears for $w_0 \rightarrow \infty$, corresponding to the plane-wave case, also discussed above.

For $w_1 \neq w_0$, one finds that (10) admits exponentially amplified solutions, which are written as $\alpha_{\pm} = \hat{\alpha}_{\pm} \exp(\lambda z)$, with the gain λ given by

$$4\lambda^2(k_y, w_1) = \frac{(w_1^2 - w_0^2)^2}{w_0^8} - \left(k_y^2 - \frac{1}{w_1^2} + \frac{1}{w_0^2}\right)^2; \quad (11)$$

with $r = w_1/w_0$, one has

$$4\lambda^2(k_y, r)w_0^4 = (r^2 - 1)^2 - [(k_y w_0)^2 + 1 - 1/r^2]^2. \quad (12)$$

The most unstable perturbation corresponds to the values r and k_y that maximize λ^2 . As detailed below, the analysis of Eq.(12) identifies two kinds of instabilities: with respect to perturbations of width greater than the pump beam ($r > 1$), and the opposite case ($r < 1$), denoted hereafter as *defocusing* and *fragmenting* instabilities, respectively.

Defocusing instability — For a perturbation with $w_1 > w_0$, the condition $\lambda^2 > 0$ predicts the maximum gain at $k_y = 0$ and given by

$$\lambda_D = \frac{\sqrt{(r^2 - 1)^2 - (1 - 1/r^2)^2}}{2w_0^2}. \quad (13)$$

λ_D is positive only for $r > 1$, and is shown in Fig.1a. This maximum gain is not limited, and grows with r , thus revealing a self-propelling instability, such that if a perturbation with waist greater than the beam is superimposed, the beam tends to spread (the perturbation gains energy) and the spreading rate increases with the waist of the beam. Note that the gain is maximum at $k_y = 0$, denoting an instability that does not tend to alter the striped shape of the beam by introducing periodic modulations. This process is also more pronounced for small waists, as the maximum gain λ_D goes like w_0^{-2} . We show in figure 2, an example of this instability, as obtained by numerically solving Eq.(3). The evolution reveals a defocusing of the beam, which is hence unstable, and is compared with the linear case $\sigma = 0$ (linear propagation); for $\sigma > 0$, the effect is more pronounced as the waist w_0 is reduced. Note that such an instability is not observed for the two dimensional scale-free beams, neither in the numerical simulations nor in the experiments,[10] and these appear to be a sort of attractor for the dynamics of the stripe beam, as shown in the following.

Fragmenting instability — For $r < 1$ (perturbation smaller than the pump, i.e., $w_1 < w_0$), the gain is maximum at a $k_y > 0$, fixed by r , and given by

$$k_{y,max} = \frac{\sqrt{1/r^2 - 1}}{w_0}. \quad (14)$$

The corresponding maximum growth rate λ_R is

$$\lambda_F = \frac{1 - r^2}{2w_0^2}. \quad (15)$$

However, as $r < 1$, the maximum gain corresponds to $r = 0$ (vanishing w_1) with diverging $k_{y,max}$, denoting the tendency of the beam to break up into very tiny spots, with no preferential spatial scale, in great contrast with the standard MI. Additionally, we note that, for a fixed r , the gain scales as the inverse squared waist, hence the more focused is the beam, the more pronounced is the instability. This is another remarkable difference with standard MI; e.g., in Kerr media as the pump power is increased, the gain grows as well; conversely in the scale-free model, the power does not affect the gain, which, on the contrary, increases when decreasing the beam spot size. Note also that the gain level for the fragmenting instability is lower than for the defocusing one and is limited by the upper value $\lambda_F(r = 0) = 1/(2w_0^2)$, longer propagation distances are needed to appreciate its development. A notable outcome is that tiny details superimposed onto the pump are amplified upon propagation. In Fig.3a,b, we show an example of the fragmenting instability, by the evolution of a stripe perturbed by a Gaussian noise with 10% amplitude with respect to the pump.

In Fig.3c, we show the spectrum obtained numerically from Eq.(3), averaged over 10 noise realizations, and

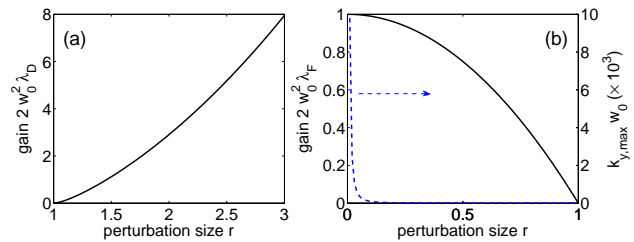


FIG. 1: (a) Gain versus the ratio of the waist between the perturbation and the pump for $r > 1$ (maximum gain attained at $k_y = 0$); (b) Left scale: gain versus r for $r < 1$; right scale: corresponding maximally amplified period.

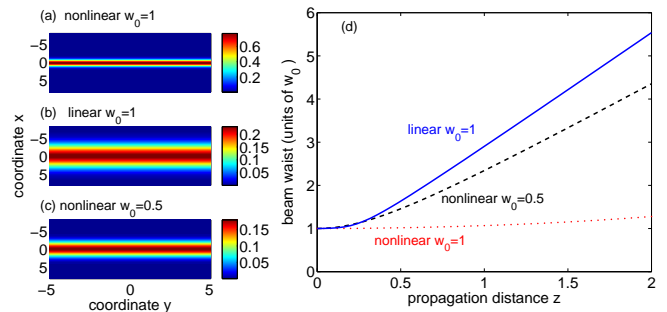


FIG. 2: (Color online) Simulation of the defocusing instability after Eq.(3). (a) Output beam at $z = 2$ for $\sigma = 0.125$, with a striped beam with $w_0 = 1$ and $A_0 = 1$; (b) as in (a) with $\sigma = 0$ (linear propagation); (c) as in (a) with $w_0 = 0.5$; (d) beam waist in the x directions for (a), dotted line, for (b), dashed line, for (c) continuous line.

compared with that expected from Eq.(12); the shaded region corresponds to the overlap of the amplified spectral regions for various r and multiplied by a Gaussian spectral content. We report spectra for an initial waist $w_0 = 0.5$ and $w_0 = 4$, showing that in the latter case the instability is moderated by about an order of magnitude, as expected from Eq.(12). Spectra are arbitrarily shifted in the vertical axis for the sake of comparison. In recent experiments in copper doped KTN:Li, the described instability has been observed for with beam waist in the x -direction of the order of $15\mu\text{m}$, after a propagation distance of 6mm , corresponding to $W_0 = 30\mu\text{m}$ and $w_0 = 0.5$ in Fig.3. [10]

Conclusions — We have investigated wave instability in systems supporting scale-free optics, such as vitrified photorefractive dipolar glass. Without a dominant spatial scale, MI is found to disappear, outdone by two new forms of instability with a number of interesting physical and mathematical properties. Scale-free instability is, in fact, able to amplify tiny beam perturbations at any spatial scale (in the paraxial approximation), without washing out or filtering the noise. The instability hence allows the wave to pick-up and carry large amounts of information, a fact that opens interesting perspectives

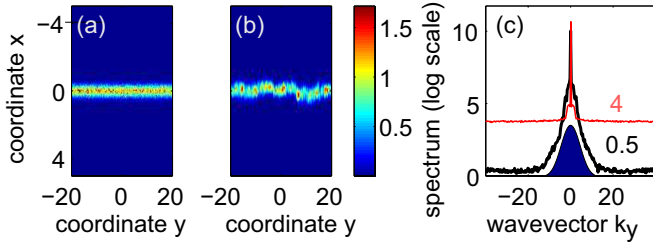


FIG. 3: (Color online) Simulation of the fragmenting instability. The stripe solution is perturbed by a Gaussian noise p with 10% amplitude ($|p/\psi_0| = 0.1$); (a) input beam; (b) output beam at $z = 2$ ($A_0 = 1, w_0 = 0.5$, $\sigma = 0.125$); (c) spectrum obtained after the numerical simulations (continuous thick line for $w_0 = 0.5$, thin line $w_0 = 4$), the colored region is after eq. (12) showing the theoretically expected spectrum.

for imaging applications.

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